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Z_3-graded differential geometry of the quantum plane

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Abstract

In this work, the Z_3-graded differential geometry of the quantum plane is constructed. The corresponding quantum Lie algebra and its Hopf algebra structure are obtained. The dual algebra, i.e. the universal enveloping algebra of the quantum plane is explicitly constructed.

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1. Introduction

After the discovery of the quantum plane by Manin [1], Wess and Zumino [2] developed a differential calculus on the quantum (hyper) plane covariant with respect to the action of the quantum group. In their method, the R-matrix is obtained using the consistency conditions. This leads to a consistent exterior derivative. The purely algebraic properties of these recently discovered spaces have been deeply discussed. The q-differential algebras have become the subject of excellent works [3, 4].

The Z_3-graded algebraic structures have been introduced by Kerner [5] and studied in [6]. Other studies on the Z_3-graded structures can be found in [7]. The de Rham complex with differential operator d satisfying the Q-Leibniz rule and the condition $\mathsf{d}^3 = 0$ on an associative unital algebra has been constructed by Bazunova *et al* [8] using the methods of [2]. This paper considers an alternative approach where, instead of adopting an R-matrix, consistency conditions on natural commutation relations are used.

The cyclic group Z_3 can be represented in the complex plane by means of the cubic roots of 1: let $j = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$ ($\mathrm{i}^2 = -1$). Then one has

$$j^3 = 1 \qquad \text{and} \qquad j^2 + j + 1 = 0 \qquad \text{or} \qquad (j+1)^2 = j. \tag{1}$$

One can define the Z_3-graded commutator $[A, B]$ as [3]

$$[A, B]_{Z_3} = AB - j^{\deg(A)\deg(B)} BA \tag{2}$$

where $\deg(X)$ denotes the grade of X. If A and B are j-commutative, then we have

$$AB = j^{\deg(A)\deg(B)} BA. \tag{3}$$

2. Review of Hopf algebra $\mathcal{A}$

The elementary properties of the extended quantum plane are described in [9]. We briefly state the properties we are going to need in this work.

2.1. The algebra of polynomials on the q-plane

The quantum plane [1] is defined as an associative algebra generated by two noncommuting coordinates x and y with the relation

$$xy - qyx = 0 \qquad q \in \mathcal{C} - \{0\}. \tag{4}$$

This associative algebra over the complex numbers, $\mathcal{C}$, is known as the algebra of polynomials over the quantum plane and is often denoted by $C_q[x, y]$. In the limit $q \longrightarrow 1$, this algebra is commutative and can be considered as the algebra of polynomials $C[x, y]$ over the usual plane, where x and y are the two coordinate functions. Below we show that a Z_3-graded *commutative* differential calculus cannot exist as in the Z_2-grade case. We denote the unital extension of C_q by $\mathcal{A}$.

2.2. Hopf algebra structure on $\mathcal{A}$

The definitions of a coproduct, a counit and a coinverse on the algebra $\mathcal{A}$ are as follows [9, 10]:

(1) The coproduct $\Delta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A} \otimes \mathcal{A}$ is defined by

$$\Delta_{\mathcal{A}}(x) = x \otimes x \qquad \Delta_{\mathcal{A}}(y) = y \otimes 1 + x \otimes y \tag{5}$$

is coassociative:

$$(\Delta_{\mathcal{A}} \otimes \mathrm{id}) \circ \Delta_{\mathcal{A}} = (\mathrm{id} \otimes \Delta_{\mathcal{A}}) \circ \Delta_{\mathcal{A}} \tag{6}$$

where id denotes the identity map on $\mathcal{A}$.

(2) The counit $\epsilon_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{C}$ is given by

$$\epsilon_{\mathcal{A}}(x) = 1 \qquad \epsilon_{\mathcal{A}}(y) = 0. \tag{7}$$

The counit $\epsilon_{\mathcal{A}}$ has the property

$$m_{\mathcal{A}} \circ (\epsilon_{\mathcal{A}} \otimes \mathrm{id}) \circ \Delta_{\mathcal{A}} = m_{\mathcal{A}} \circ (\mathrm{id} \otimes \epsilon_{\mathcal{A}}) \circ \Delta_{\mathcal{A}} \tag{8}$$

where $m_{\mathcal{A}}$ stands for the algebra product $\mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$.

(3) If we extend the algebra $\mathcal{A}$ by adding the inverse of x then the algebra $\mathcal{A}$ admits a $\mathcal{C}$-algebra antihomomorphism (coinverse) $\kappa_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$\kappa_{\mathcal{A}}(x) = x^{-1} \qquad \kappa_{\mathcal{A}}(y) = -x^{-1}y. \tag{9}$$

The coinverse $\kappa_{\mathcal{A}}$ satisfies

$$m_{\mathcal{A}} \circ (\kappa_{\mathcal{A}} \otimes \mathrm{id}) \circ \Delta_{\mathcal{A}} = \epsilon_{\mathcal{A}} = m_{\mathcal{A}} \circ (\mathrm{id} \otimes \kappa_{\mathcal{A}}) \circ \Delta_{\mathcal{A}}. \tag{10}$$

3. Construction of bicovariant Z_3-graded differential calculus on $\mathcal{A}$

The Woronowicz theory [11] is based on the idea that the differential and algebraic structures of $\mathcal{A}$ can coact covariantly on the algebra of its differential calculus over $\mathcal{A}$. We first recall some basic notions about differential calculus on the extended q-plane.

3.1. Differential algebra

To begin with, we note the properties of the exterior differential d. The exterior differential d is an operator which gives the mapping from the generators of $\mathcal{A}$ to the differentials

$$\mathsf{d} : a \longrightarrow \mathsf{d}a \qquad a \in \{x, y\}.$$

We require that the exterior differential d has to satisfy two properties

$$\mathsf{d}^3 = 0 \tag{11}$$

and the Z_3-graded Leibniz rule

$$\mathsf{d}(fg) = (\mathsf{d}f)g + j^{\deg(f)}(\mathsf{d}g). \tag{12}$$

In order to establish a noncommutative differential calculus including second-order differentials of the generators of $\mathcal{A}$ on the q-plane, we assume that the commutation relations between the coordinates and their first-order differentials are of the following form:

$$\begin{aligned} x\,\mathsf{d}x &= A\,\mathsf{d}x\,x \\ x\,\mathsf{d}y &= C_{11}\,\mathsf{d}y\,x + C_{12}\,\mathsf{d}x\,y \\ y\,\mathsf{d}x &= C_{21}\,\mathsf{d}x\,y + C_{22}\,\mathsf{d}y\,x \\ y\,\mathsf{d}y &= B\,\mathsf{d}y\,y. \end{aligned} \tag{13}$$

The coefficients A, B and C_{ik} will be determined in terms of the complex deformation parameter q and j. To find them we shall use the covariance of the noncommutative differential calculus.

Since we assume that $\mathsf{d}^3 = 0$ and $\mathsf{d}^2 \neq 0$, in order to construct a self-consistent theory of differential forms it is necessary to add to the first-order differentials of coordinates $\mathsf{d}x$, $\mathsf{d}y$ a set of *second-order differentials* d^2x, d^2y. Let us begin by assuming that

$$\mathsf{d}x\,\mathsf{d}y = F\,\mathsf{d}y\,\mathsf{d}x \qquad (\mathsf{d}x)^3 = 0 = (\mathsf{d}y)^3 \tag{14}$$

where F is a parameter that will be specified later on.

The first differentiation of (13) gives rise to the relations between the generators x, y and the second-order differentials d^2x, d^2y including first-order differentials

$$\begin{aligned} x\,\mathsf{d}^2x &= A\,\mathsf{d}^2x\,x + (Aj-1)(\mathsf{d}x)^2 \\ x\,\mathsf{d}^2y &= C_{11}\,\mathsf{d}^2y\,x + C_{12}\,\mathsf{d}^2x\,y + K_1\,\mathsf{d}y\,\mathsf{d}x \\ y\,\mathsf{d}^2x &= C_{21}\,\mathsf{d}^2x\,y + C_{22}\,\mathsf{d}^2y\,x + K_2\,\mathsf{d}y\,\mathsf{d}x \\ y\,\mathsf{d}^2y &= B\,\mathsf{d}^2y\,y + (Bj-1)(\mathsf{d}y)^2 \end{aligned} \tag{15}$$

where

$$K_1 = jC_{11} + jC_{12}F - F \qquad K_2 = jC_{21}F + jC_{22} - 1. \tag{16}$$

The relations (15) are *not homogeneous* in the sense that the commutation relations between the generators and second-order differentials include first-order differentials as well. In the following subsection, we shall see that the commutation relations between the coordinates and their second-order differentials can be made homogeneous. They will not include first-order differentials by removing them using the covariance of the noncommutative differential calculus.

Applying the exterior differential d to the relations (15), we get

$$\begin{aligned} \mathsf{d}x\,\mathsf{d}^2x &= j^{-2}\,\mathsf{d}^2x\,\mathsf{d}x \\ \mathsf{d}x\,\mathsf{d}^2y &= j^2C_{11}Q_1^{-1}\,\mathsf{d}^2y\,\mathsf{d}x + (j^2C_{12} + F^{-1}K_1)Q_1^{-1}\,\mathsf{d}^2x\,\mathsf{d}y \\ \mathsf{d}y\,\mathsf{d}^2x &= j^2C_{21}Q_2^{-1}\,\mathsf{d}^2x\,\mathsf{d}y + (j^2C_{22} + K_2)Q_2^{-1}\,\mathsf{d}^2y\,\mathsf{d}x \\ \mathsf{d}y\,\mathsf{d}^2y &= j^{-2}\,\mathsf{d}^2y\,\mathsf{d}y \end{aligned} \tag{17}$$

where

$$Q_1 = -j^2(C_{12} + C_{11}F^{-1} + 1) \qquad Q_2 = -j^2(C_{22} + C_{21}F + 1). \tag{18}$$

The differentiation of second or third relations of (17) gives rise to the relations between the second-order differentials:

$$\mathrm{d}^2x\,\mathrm{d}^2y = F\,\mathrm{d}^2y\,\mathrm{d}^2x. \tag{19}$$

3.2. Covariance

In order to homogenize the relations (15), we shall consider the covariance of the noncommutative differential calculus. Let Γ be a bimodule over the algebra $\mathcal{A}$ generated by the elements of the set $\{x, y, \mathrm{d}x, \mathrm{d}y, \mathrm{d}^2x, \mathrm{d}^2y\}$. One says that (Γ, d) is a *first-order differential calculus* over the Hopf algebra $(\mathcal{A}, \Delta_{\mathcal{A}}, \epsilon_{\mathcal{A}}, \kappa_{\mathcal{A}})$. We start with the definitions of a left- and right-covariant bimodule.

(1) Let Γ be a bimodule over $\mathcal{A}$ and $\Delta^R : \Gamma \longrightarrow \Gamma \otimes \mathcal{A}$ be a linear homomorphism. We say that (Γ, Δ^R) is a right-covariant bimodule if

$$\Delta^R(a\rho + \rho' a') = \Delta_{\mathcal{A}}(a)\Delta^R(\rho) + \Delta^R(\rho')\Delta_{\mathcal{A}}(a') \tag{20}$$

for all $a, a' \in \mathcal{A}$ and $\rho, \rho' \in \Gamma$, and

$$(\Delta^R \otimes \mathrm{id}) \circ \Delta^R = (\mathrm{id} \otimes \Delta_{\mathcal{A}}) \circ \Delta^R \qquad (\mathrm{id} \otimes \epsilon) \circ \Delta^R = \mathrm{id}. \tag{21}$$

The action of Δ^R on the first-order differentials is

$$\Delta^R(\mathrm{d}x) = \mathrm{d}x \otimes x \qquad \Delta^R(\mathrm{d}y) = \mathrm{d}y \otimes 1 + \mathrm{d}x \otimes y \tag{22}$$

since

$$\Delta^R(\mathrm{d}a) = (\mathrm{d} \otimes \mathrm{id})\Delta_{\mathcal{A}}(a) \qquad \forall a \in \mathcal{A}. \tag{23}$$

We now apply the linear map Δ^R to relations (13)

$$\begin{aligned}
\Delta^R(x\,\mathrm{d}x) &= \Delta_{\mathcal{A}}(x)\Delta^R(\mathrm{d}x) = A\Delta^R(\mathrm{d}x\,x)\\
\Delta^R(x\,\mathrm{d}y) &= C_{11}\Delta^R(\mathrm{d}y\,x) + C_{12}\Delta^R(\mathrm{d}x\,y) + (qA - C_{11} - qC_{12})\,\mathrm{d}x\,x \otimes xy\\
\Delta^R(y\,\mathrm{d}x) &= C_{21}\Delta^R(\mathrm{d}x\,y) + C_{22}\Delta^R(\mathrm{d}y\,x) + (A - qC_{21} - C_{22})\,\mathrm{d}x\,x \otimes yx\\
\Delta^R(y\,\mathrm{d}y) &= B\Delta^R(\mathrm{d}y\,y) + (C_{12} + C_{21} - B)\,\mathrm{d}x\,y \otimes y + (A - B)\,\mathrm{d}x\,x \otimes y^2\\
&\quad + (C_{11} + C_{22} - B)\,\mathrm{d}y\,x \otimes y,
\end{aligned}$$

and relations (14)

$$\Delta^R(\mathrm{d}x\,\mathrm{d}y) = F\Delta^R(\mathrm{d}y\,\mathrm{d}x) + (q - F)(\mathrm{d}x)^2 \otimes yx.$$

So we must have

$$\begin{aligned}
&C_{11} + qC_{12} = qA \qquad && C_{11} + C_{22} = B \qquad && A = B\\
&qC_{21} + C_{22} = A && C_{12} + C_{21} = B && F = q.
\end{aligned} \tag{24}$$

(2) Let Γ be a bimodule over $\mathcal{A}$ and $\Delta^L : \Gamma \longrightarrow \mathcal{A} \otimes \Gamma$ be a linear homomorphism. We say that (Γ, Δ^L) is a left-covariant bimodule if

$$\Delta^L(a\rho + \rho' a') = \Delta_{\mathcal{A}}(a)\Delta^L(\rho) + \Delta^L(\rho')\Delta_{\mathcal{A}}(a') \tag{25}$$

for all $a, a' \in \mathcal{A}$ and $\rho, \rho' \in \Gamma$, and

$$(\Delta_{\mathcal{A}} \otimes \mathrm{id}) \circ \Delta^L = (\mathrm{id} \otimes \Delta^L) \circ \Delta^L \qquad (\epsilon \otimes \mathrm{id}) \circ \Delta^L = \mathrm{id}. \tag{26}$$

Since

$$\Delta^L(\mathrm{d}a) = (\mathrm{id} \otimes \mathrm{d})\Delta_{\mathcal{A}}(a) \qquad \forall a \in \mathcal{A} \tag{27}$$

the action of Δ^L on the first-order differentials gives rise to the relations

$$\Delta^L(\mathrm{d}x) = x \otimes \mathrm{d}x \qquad \Delta^L(\mathrm{d}y) = x \otimes \mathrm{d}y. \tag{28}$$

Applying Δ^L to relations (13), we get

$$C_{12} = 0 \qquad C_{21} = q^{-1} \qquad B = q^{-1}. \tag{29}$$

With the relations (24), we then obtain

$$\begin{array}{lll} A = q^{-1} & C_{11} = 1 & C_{21} = q^{-1} \\ B = q^{-1} & C_{12} = 0 & C_{22} = q^{-1} - 1. \end{array} \tag{30}$$

So

$$K_1 = j - q \qquad K_2 = q^{-1}(j-q) \qquad Q_1 = -j^2(q^{-1}+1) = Q_2.$$

On the other hand, since the differential of a function f of the coordinates x and y is of the form

$$\mathrm{d}f = (\mathrm{d}x\,\partial_x + \mathrm{d}y\,\partial_y) f \tag{31}$$

and

$$\begin{aligned} \mathrm{d}^2 f &= \left(\mathrm{d}^2x\,\partial_x + \mathrm{d}^2y\,\partial_y + j(\mathrm{d}x)^2\partial_x^2 + j(\mathrm{d}y)^2\partial_y^2 + \mathrm{d}x\,\mathrm{d}y(\partial_x\partial_y + q\partial_y\partial_x)\right) f \\ \mathrm{d}^3 f &= \mathrm{d}^2x\,\mathrm{d}y\left(j^2\partial_y\partial_x + q^{-1}j\partial_x\partial_y + \frac{1-jq^{-1}}{q+1}\partial_x\partial_y + \frac{q-j}{q+1}\partial_y\partial_x\right) f \\ &\quad + \mathrm{d}^2y\,\mathrm{d}x\left(j^2\partial_x\partial_y - \frac{j^2}{q+1}\partial_x\partial_y - \frac{1}{q+1}\partial_y\partial_x\right) f + \cdots \\ &= \frac{j^2}{q+1}\mathrm{d}^2x\,\mathrm{d}y(\partial_y\partial_x - \partial_x\partial_y) + \frac{1}{q+1}\mathrm{d}^2y\,\mathrm{d}x(qj^2\partial_x\partial_y - \partial_y\partial_x) f + \cdots \\ &\equiv 0 \end{aligned}$$

we have

$$\partial_x\partial_y = \partial_y\partial_x \tag{32}$$

if q satisfies the identities

$$qj^2 = 1 \qquad q^2 + q + 1 = 0. \tag{33}$$

One can then choose

$$q = j^{-2} = j. \tag{34}$$

Consequently, the relations (13)–(15), (17) and (19) are explicitly as follows: the commutation relations between the coordinates and their first-order differentials are [12]

$$\begin{array}{ll} x\,\mathrm{d}x = q^{-1}\,\mathrm{d}x\,x & x\,\mathrm{d}y = \mathrm{d}y\,x \\ y\,\mathrm{d}x = q^{-1}\,\mathrm{d}x\,y + (q^{-1}-1)\,\mathrm{d}y\,x & y\,\mathrm{d}y = q^{-1}\,\mathrm{d}y\,y \end{array} \tag{35}$$

and among those first-order differentials are

$$\mathrm{d}x\,\mathrm{d}y = q\,\mathrm{d}y\,\mathrm{d}x \qquad (\mathrm{d}x)^3 = 0 = (\mathrm{d}y)^3. \tag{36}$$

The commutation relations between variables and second-order differentials are

$$\begin{array}{ll} x\,\mathrm{d}^2x = q^{-1}\,\mathrm{d}^2x\,x & x\,\mathrm{d}^2y = \mathrm{d}^2y\,x \\ y\,\mathrm{d}^2y = q^{-1}\,\mathrm{d}^2y\,y & y\,\mathrm{d}^2x = q^{-1}\,\mathrm{d}^2x\,y + (q^{-1}-1)\,\mathrm{d}^2y\,x. \end{array} \tag{37}$$

The commutation relations between first-order and second-order differentials are

$$\begin{aligned} &\mathrm{d}x\,\mathrm{d}^2x = q^{-2}\,\mathrm{d}^2x\,\mathrm{d}x && \mathrm{d}x\,\mathrm{d}^2y = q^2\,\mathrm{d}^2y\,\mathrm{d}x \\ &\mathrm{d}y\,\mathrm{d}^2y = q^{-2}\,\mathrm{d}^2y\,\mathrm{d}y && \mathrm{d}y\,\mathrm{d}^2x = q^{-2}\,\mathrm{d}^2x\,\mathrm{d}y + (q - q^{-1})\,\mathrm{d}^2y\,\mathrm{d}x \end{aligned} \tag{38}$$

and those among the second-order differentials are

$$\mathrm{d}^2x\,\mathrm{d}^2y = q\,\mathrm{d}^2y\,\mathrm{d}^2x. \tag{39}$$

Now, it can be checked that the linear maps Δ^R and Δ^L leave invariant the relations (35)–(39). One can also check that the identities (21), (26) and also the following identities are satisfied:

$$(\mathrm{id} \otimes \mathrm{d})\Delta_{\mathcal{A}}(a) = \Delta^L(\mathrm{d}a) \qquad (\mathrm{d} \otimes \mathrm{id})\Delta_{\mathcal{A}}(a) = \Delta^R(\mathrm{d}a) \tag{40}$$

and

$$(\Delta^L \otimes \mathrm{id}) \circ \Delta^R = (\mathrm{id} \otimes \Delta^R) \circ \Delta^L. \tag{41}$$

4. Cartan–Maurer one-forms on $\mathcal{A}$

In analogy with the left-invariant one-forms on a Lie group in classical differential geometry, one can construct two one-forms using the generators of $\mathcal{A}$ as follows [9]:

$$\theta = \mathrm{d}x\,x^{-1} \qquad \varphi = \mathrm{d}y - \mathrm{d}x\,x^{-1}y. \tag{42}$$

The commutation relations between the generators of $\mathcal{A}$ and one-forms are [9]

$$\begin{aligned} &x\theta = q^{-1}\theta x && y\theta = q^{-1}\theta y + (q^{-1} - 1)\varphi \\ &x\varphi = \varphi x && y\varphi = \varphi y. \end{aligned} \tag{43}$$

The first-order differentials with one-forms satisfy the following relations

$$\begin{aligned} &\theta\,\mathrm{d}x = q\,\mathrm{d}x\,\theta && \varphi\,\mathrm{d}x = \mathrm{d}x\,\varphi \\ &\theta\,\mathrm{d}y = q\,\mathrm{d}y\,\theta && \varphi\,\mathrm{d}y = \mathrm{d}y\,\varphi \end{aligned} \tag{44}$$

and with second-order differentials

$$\begin{aligned} &\theta\,\mathrm{d}^2x = q^2\,\mathrm{d}^2x\,\theta && \theta\,\mathrm{d}^2x = q^2\,\mathrm{d}^2x\,\theta \\ &\varphi\,\mathrm{d}^2x = q^{-2}\,\mathrm{d}^2x\,\varphi && \varphi\,\mathrm{d}^2y = q^{-2}\,\mathrm{d}^2y\,\varphi. \end{aligned} \tag{45}$$

The commutation rules of the elements θ and φ are

$$\theta^3 = 0 \qquad \theta\varphi = \varphi\theta \tag{46a}$$

and

$$\varphi^3 = 0 \tag{46b}$$

provided that $q^2 + q + 1 = 0$.

We denote the algebra of the forms generated by the two elements θ and φ by Ω. We make the algebra Ω into a Z_3-graded Hopf algebra with the following co-structures [9]: the coproduct $\Delta_\Omega : \Omega \longrightarrow \Omega \otimes \Omega$ is defined by

$$\Delta_\Omega(\theta) = \theta \otimes 1 + 1 \otimes \theta \qquad \Delta_\Omega(\varphi) = \varphi \otimes 1 + x \otimes \varphi - y \otimes \theta. \tag{47}$$

The counit $\epsilon_\Omega : \Omega \longrightarrow \mathcal{C}$ is given by

$$\epsilon_\Omega(\theta) = 0 \qquad \epsilon_\Omega(\varphi) = 0 \tag{48}$$

and the coinverse $\kappa_\Omega : \Omega \longrightarrow \Omega$ is defined by

$$\kappa_\Omega(\theta) = -\theta \qquad \kappa_\Omega(\varphi) = -q^{-1}\varphi x^{-1} - \theta x^{-1}y. \tag{49}$$

One can easily check that (6), (8) and (10) are satisfied. Note that the commutation relations (43)–(46) are compatible with Δ_Ω, ϵ_Ω and κ_Ω, in the sense that $\Delta_\Omega(x\theta) = q^{-1}\Delta_\Omega(\theta x)$, and so on.

5. Quantum Lie algebra

The commutation relations of Cartan–Maurer forms allow us to construct the algebra of the generators. In order to obtain the quantum Lie algebra of the algebra generators we first write the Cartan–Maurer forms as

$$\mathsf{d}x = \theta x \qquad \mathsf{d}y = \varphi + \theta y. \tag{50}$$

The differential d can then be expressed in the form

$$\mathsf{d} = \theta H + \varphi X. \tag{51}$$

Here H and X are the quantum Lie algebra generators. We shall now obtain the commutation relations of these generators. Considering an arbitrary function f of the coordinates of the quantum plane and using that $\mathsf{d}^3 = 0$ one has

$$\mathsf{d}^2 f = \mathsf{d}\theta Hf + \mathsf{d}\varphi Xf + j\theta\,\mathsf{d}Hf + j\varphi\,\mathsf{d}Xf$$

and

$$\mathsf{d}^3 f = \mathsf{d}^2\theta Hf + \mathsf{d}^2\varphi Xf + j^2\,\mathsf{d}\theta\,\mathsf{d}Hf + j^2\,\mathsf{d}\varphi\,\mathsf{d}Xf + j^2\theta\,\mathsf{d}^2Hf + j^2\varphi\,\mathsf{d}^2Xf.$$

So we need the two-forms. Applying the exterior differential d to the relations (42) one has

$$\begin{aligned}\mathsf{d}\theta &= \mathsf{d}^2x\,x^{-1} - j\theta^2\\ \mathsf{d}\varphi &= \mathsf{d}^2y - \mathsf{d}^2x\,x^{-1}y - j\theta\varphi.\end{aligned} \tag{52}$$

Also, since

$$\begin{aligned}\theta\,\mathsf{d}\theta &= q^{-2}\,\mathsf{d}\theta\,\theta\\ \theta\,\mathsf{d}\varphi &= q^2\,\mathsf{d}\varphi\,\theta + (q - q^{-1})\,\mathsf{d}\theta\,\varphi + (q^{-1} - q)\theta^2\varphi\\ \varphi\,\mathsf{d}\theta &= q^{-2}\,\mathsf{d}\theta\,\varphi + (q^{-1} - q)\theta^2\varphi\\ \varphi\,\mathsf{d}\varphi &= q^{-2}\,\mathsf{d}\varphi\,\varphi + (q^{-1} - q)\theta\varphi^2\end{aligned} \tag{53}$$

we have

$$\mathsf{d}^2\theta = 0 \qquad \mathsf{d}^2\varphi = j\,\mathsf{d}\theta\,\varphi - j\,\mathsf{d}\varphi\,\theta - j\theta^2\varphi. \tag{54}$$

Using the Cartan–Maurer equations we find the following commutation relation for the quantum Lie algebra

$$XH = q^{-1}HX + X. \tag{55}$$

The commutation relation (55) of the algebra generators should be consistent with the monomials of the coordinates of the quantum plane. To do this, we evaluate the commutation relations between the generators of algebra and the coordinates. The commutation relations between the generators and the coordinates can be extracted from the Z_3-graded Leibniz rule

$$\begin{aligned}\mathsf{d}(xf) &= (\mathsf{d}x)f + x(\mathsf{d}f)\\ &= \theta(x + q^{-1}xH)f + \varphi(xX)f\\ &= (\theta H + \varphi X)xf\end{aligned} \tag{56}$$

and

$$\begin{aligned}\mathsf{d}(yf) &= (\mathsf{d}y)f + y(\mathsf{d}f)\\ &= \theta(y + q^{-1}yH)f + \varphi(1 + yX + (q^{-1} - 1)H)f\\ &= (\theta H + \varphi X)yf.\end{aligned} \tag{57}$$

This yields

$$Hx = x + q^{-1}xH \qquad Hy = y + q^{-1}yH$$
$$Xx = xX \qquad Xy = 1 + yX + (q^{-1} - 1)H. \tag{58}$$

We know that the differential operator d satisfies the Z_3-graded Leibniz rule. Therefore, the generators H and X are endowed with a natural coproduct. To find them, we need the following commutation relation

$$Hx^m = \frac{1 - q^{-m}}{1 - q^{-1}}x^m + q^{-m}x^m H \tag{59a}$$

and

$$Hy^n = \frac{1 - q^{-n}}{1 - q^{-1}}y^n + q^{-n}y^n H \tag{59b}$$

where (58) was used. The relation (59*a*) is understood as an operator equation. This implies that when H acts on arbitrary monomials $x^m y^n$,

$$H(x^m y^n) = \frac{1 - q^{-(m+n)}}{1 - q^{-1}}(x^m y^n) + q^{-(m+n)}(x^m y^n)H \tag{60}$$

from which we obtain

$$H = \frac{1 - q^{-N}}{1 - q^{-1}} \tag{61}$$

where N is a number operator acting on a monomial as

$$N(x^m y^n) = (m + n)x^m y^n. \tag{62}$$

We also have

$$X(x^m y^n) = (x^m y^n)X + \frac{1 - q^{-n}}{1 - q^{-1}}x^m y^{n-1}(1 + (q^{-1} - 1)H). \tag{63}$$

So, applying the Z_3-graded Leibniz rule to the product of functions f and g, we write

$$\mathsf{d}(fg) = [(\theta H + \varphi X)f]g + f(\theta H + \varphi X)g \tag{64}$$

with the help of (51). From the commutation relations of the Cartan–Maurer forms with the coordinates of the quantum plane, we can compute the corresponding relations of θ and φ with functions of the coordinates. From (43) we have

$$(x^m y^n)\theta = q^{-(m+n)}\theta(x^m y^n) + (q^{-n} - 1)\varphi x^m y^{n-1} \qquad (x^m y^n)\varphi = \varphi(x^m y^n). \tag{65}$$

Inserting (65) into (64) and equating coefficients of the Cartan–Maurer forms, we get, for example,

$$H(fg) = (Hf)g + q^{-N}f(Hg). \tag{66}$$

Consequently, we have the coproduct

$$\Delta(H) = H \otimes 1 + q^{-N} \otimes H$$
$$\Delta(X) = X \otimes 1 + 1 \otimes X + (q^{-1} - 1)X \otimes H. \tag{67}$$

The counit and coinverse may be calculated by using the axioms of Hopf algebra

$$m(\epsilon \otimes \mathrm{id})\Delta(u) = u \qquad m(\mathrm{id} \otimes \kappa)\Delta(u) = \epsilon(u). \tag{68}$$

So we have

$$\epsilon(H) = 0 = \epsilon(X) \tag{69}$$

$$\kappa(H) = -q^N H \qquad \kappa(X) = -Xq^N. \tag{70}$$

6. The dual of the Hopf algebra $\mathcal{A}$

In this section, in order to obtain the dual of the Hopf algebra $\mathcal{A}$ defined in section 2, we shall use the method of [13].

A pairing between two vector spaces $\mathcal{U}$ and $\mathcal{A}$ is a bilinear mapping $\langle,\rangle : \mathcal{U} \times \mathcal{A} \longrightarrow \mathcal{C}$, $(u, a) \mapsto \langle u, a\rangle$. We say that the pairing is non-degenerate if

$$\langle u, a\rangle = 0 (\forall a \in \mathcal{A}) \Longrightarrow u = 0$$

and

$$\langle u, a\rangle = 0 (\forall u \in \mathcal{U}) \Longrightarrow a = 0.$$

Such a pairing can be extended to a pairing of $\mathcal{U} \otimes \mathcal{U}$ and $\mathcal{A} \otimes \mathcal{A}$ by

$$\langle u \otimes v, a \otimes b\rangle = \langle u, a\rangle\langle v, b\rangle.$$

Given bialgebras $\mathcal{U}$ and $\mathcal{A}$ and a non-degenerate pairing

$$\langle,\rangle : \mathcal{U} \times \mathcal{A} \longrightarrow \mathcal{C} \qquad (u, a) \mapsto \langle u, a\rangle \qquad \forall u \in \mathcal{U} \quad \forall a \in \mathcal{A} \tag{71}$$

we say that the bilinear form realizes a duality between $\mathcal{U}$ and $\mathcal{A}$, or that the bialgebras $\mathcal{U}$ and $\mathcal{A}$ are in duality, if we have

$$\begin{aligned} \langle uv, a\rangle &= \langle u \otimes v, \Delta_{\mathcal{A}}\rangle \\ \langle u, ab\rangle &= \langle \Delta_{\mathcal{U}}(u), a \otimes b\rangle \\ \langle 1_{\mathcal{U}}, a\rangle &= \epsilon_{\mathcal{A}}(a) \\ \langle u, 1_{\mathcal{A}}\rangle &= \epsilon_{\mathcal{U}}(u) \end{aligned} \tag{72}$$

for all $u, v \in \mathcal{U}$ and $a, b \in \mathcal{A}$.

If, in addition, $\mathcal{U}$ and $\mathcal{A}$ are Hopf algebras with coinverse κ, then they are said to be in duality if the underlying bialgebras are in duality and if, moreover, we have

$$\langle \kappa_{\mathcal{U}}(u), a\rangle = \langle u, \kappa_{\mathcal{A}}(a)\rangle \qquad \forall u \in \mathcal{U} \quad a \in \mathcal{A}. \tag{73}$$

It suffices to define the pairing (71) between the generating elements of the two algebras. Pairing for any other elements of $\mathcal{U}$ and $\mathcal{A}$ follows from relations (72) and the bilinear form inherited by the tensor product. For example, for

$$\Delta_{\mathcal{U}}(u) = \sum_k u'_k \otimes u''_k$$

we have

$$\langle u, ab\rangle = \langle \Delta_{\mathcal{U}}(u), a \otimes b\rangle = \sum_k \langle u'_k, a\rangle\langle u''_k, b\rangle.$$

As a Hopf algebra $\mathcal{A}$ is generated by the elements x, y and a basis is given by all monomials of the form

$$f = x^m y^n$$

where $m, n \in \mathcal{Z}_+$. Let us denote the dual algebra by $\mathcal{U}_q$ and its generating elements by A and B.

The pairing is defined through the tangent vectors as follows:

$$\begin{aligned} \langle A, f\rangle &= m\delta_{n,0} \\ \langle B, f\rangle &= \delta_{n,1}. \end{aligned} \tag{74}$$

We also have

$$\langle 1_{\mathcal{U}}, f\rangle = \epsilon_{\mathcal{A}}(f) = \delta_{n,0}. \tag{75}$$

Using the defining relations one gets

$$\langle AB, f\rangle = (m+1)\delta_{n,1} \tag{75a}$$

and

$$\langle BA, f\rangle = m\delta_{n,1} \tag{75b}$$

where differentiation is from the right as this is most suitable for differentiation in this basis. Thus one obtains the commutation relation in the algebra $\mathcal{U}_q$ dual to $\mathcal{A}$ as

$$AB = BA + B. \tag{76}$$

The Hopf algebra structure of this algebra can be deduced by using the duality. The coproduct of the elements of the dual algebra is given by

$$\begin{aligned} \Delta_{\mathcal{U}}(A) &= A \otimes 1_{\mathcal{U}} + 1_{\mathcal{U}} \otimes A \\ \Delta_{\mathcal{U}}(B) &= B \otimes q^{A} + 1_{\mathcal{U}} \otimes B. \end{aligned} \tag{77}$$

The counity is given by

$$\epsilon_{\mathcal{U}}(A) = 0 \qquad \epsilon_{\mathcal{U}}(B) = 0. \tag{78}$$

The coinverse is given as

$$\kappa_{\mathcal{U}}(A) = -A \qquad \kappa_{\mathcal{U}}(B) = -Bq^{-A}. \tag{79}$$

We can now transform this algebra to the form obtained in section 5 by making the following definitions:

$$H = \frac{1_{\mathcal{U}} - q^{A}}{1 - q^{-1}} \qquad X = B \tag{80}$$

which are consistent with the commutation relation and the Hopf structures.

7. Conclusion

To conclude, here we introduce the commutation relations between the coordinates of the quantum plane and their partial derivatives and thus illustrate the connection between the relations in section 5, and the relations which will now be obtained.

To proceed, let us obtain the relations of the coordinates with their partial derivatives. We know that the exterior differential d can be expressed in the form

$$\mathsf{d}f = (\mathsf{d}x\,\partial_x + \mathsf{d}y\,\partial_y)f. \tag{81}$$

Then, for example,

$$\begin{aligned} \mathsf{d}(xf) &= \mathsf{d}x\, f + x\,\mathsf{d}f \\ &= \mathsf{d}x(1 + q^{-1}x\partial_x)f + \mathsf{d}y\, x\partial_y f \\ &= (\mathsf{d}x\,\partial_x x + \mathsf{d}y\,\partial_y x)f \end{aligned}$$

so that

$$\begin{aligned} \partial_x x &= 1 + q^{-1}x\partial_x & \partial_x y &= q^{-1}y\partial_x \\ \partial_y x &= x\partial_y & \partial_y y &= 1 + q^{-1}y\partial_y + (q^{-1} - 1)x\partial_x. \end{aligned} \tag{82}$$

The Hopf algebra structure for ∂ is given by

$$\begin{aligned} \Delta(\partial_x) &= \partial_x \otimes \partial_x & \Delta(\partial_y) &= \partial_y \otimes 1 + \partial_x \otimes \partial_y \\ \epsilon(\partial_x) &= 1 & \epsilon(\partial_y) &= 0 \\ \kappa(\partial_x) &= \partial_x^{-1} & \kappa(\partial_y) &= -\partial_x^{-1}\partial_y \end{aligned} \tag{83}$$

provided that the formal inverse ∂_x^{-1} exists.

We know from section 5 that the exterior differential d can be expressed in the form (51), which we repeat here

$$\mathsf{d}f = (\theta H + \varphi X) f. \tag{84}$$

Considering (81) together with (84) and using (50) one has

$$H \equiv x\partial_x + y\partial_y \qquad X \equiv \partial_y. \tag{85}$$

Using the relations (82) and (32) one can check that the relation between the generators in (85) coincides with (55). It can also be verified that the action of the generators in (85) on the coordinates coincides with (58).

We finally introduce complex notation with a single variable $z = x + \mathrm{i}y$ where x and y are the generators of the q-plane and $\mathrm{i}^2 = -1$. Then the elements

$$z \qquad \bar{z} = x - \mathrm{i}y \qquad \mathsf{d}z = \mathsf{d}x + \mathrm{i}\,\mathsf{d}y \qquad \mathsf{d}\bar{z} = \mathsf{d}x - \mathrm{i}\,\mathsf{d}y \tag{86}$$

form the basis in the algebra Γ. These elements obey the following commutation relations:

$$\begin{aligned} &z\,\mathsf{d}z = q^{-1}\,\mathsf{d}z\,z && (\bar{z}\,\mathsf{d}\bar{z} = q^{-1}\,\mathsf{d}\bar{z}\,\bar{z}) \\ &z\,\mathsf{d}^2 z = q^{-1}\,\mathsf{d}^2 z\,z && (\bar{z}\,\mathsf{d}^2\bar{z} = q^{-1}\,\mathsf{d}^2\bar{z}\,\bar{z}) \\ &\mathsf{d}z\,\mathsf{d}^2 z = q^{-2}\,\mathsf{d}^2 z\,\mathsf{d}z && (\mathsf{d}\bar{z}\,\mathsf{d}^2\bar{z} = q^{-2}\,\mathsf{d}^2\bar{z}\,\mathsf{d}\bar{z}) \\ &(\mathsf{d}z)^3 = 0 = (\mathsf{d}\bar{z})^3. \end{aligned} \tag{87}$$

Note that these relations are the same as those of [14] except that in our case z^3 need not be zero.

The Z_3-graded noncommutative differential geometry we have constructed satisfies all expectations for such a structure. In particular, all Hopf algebra axioms are satisfied without any modification.

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References

[1] Manin Yu I 1988 Quantum groups and noncommutative geometry *Montreal University Preprint* CRM-1561
[2] Wess J and Zumino B 1990 *Nucl. Phys.* B **18** 302
[3] Kerner R 1996 *Lett. Math. Phys.* **36** 441
[4] Cerchiai B, Hinterding R, Madore J and Wess J 1998 *Eur. Phys. J.* **8** 547
[5] Kerner R 1992 *J. Math. Phys.* **33** 403
[6] Kerner R 2000 *Preprint* math-ph/0011023
Kerner R 1997 *Class. Quantum Grav.* **14** A203
Plyushchay M S and Rausch de Traubenberg M 2000 *Phys. Lett.* B **477** 276
[7] Dubois-Violette M and Kerner R 1996 *Acta Math. Univ. Comenianiae LXV* **2** 175
Dubois-Violette M and Kerner R 1997 *J. Geo. Phys.* **23** 235
Dubois-Violette M and Kerner R 1997 *Preprint* q-alg/9710021
[8] Bazunova N, Borowiec A and Kerner R 2000 *Preprint* math-ph/0110007
[9] Celik S A and Celik S 2000 *Int. J. Mod. Phys.* A **15** 3237
[10] Brzezinski T 1993 *Lett. Math. Phys.* **27** 287
Brzezinski T and Majid S 1992 *Lett. Math. Phys.* **26** 67

[11] Woronowicz S L 1989 *Commun. Math. Phys.* **122** 125
[12] Brzezinski T, Dabrowski H and Rembielinski J 1992 *J. Math. Phys.* **33** 19
[13] Sudbery A 1990 *Proc. Workshop on Quantum Groups (Argonne)* ed T Curtright, D Fairlie and C Zachos (Singapore: World Scientific) pp 33–51
Dobrev V K 1992 *J. Math. Phys.* **33** 3419
[14] Abramov V and Bazunova N 2000 *Preprint* math-ph/0001041